Ch2- Particle Kineamtics 3

Normal-Tangential coordinate systems and relative motion

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Normal-Tangential coordinate systems

In some applications its more convinient to express position as a function of velocity along a path,

- Racing a car around a track
- Local path planning

Normal-Tangential coordinate systems

- Velocity is always tangent to the path, so choose velocity as one direction
- Normal is the vector from the particle's location to center of curvature.

<img src = 'images/NT.png'>

Velocity vector,

As velocity is always tangent to the path,

$$\vec{v} = v\hat{u}_t$$

Acceleration

$$\vec{a} = \frac{d\vec{v}}{dt}$$
$$\vec{a} = \dot{v}\hat{u}_t + v\dot{\hat{u}}_t$$

Computing $\dot{\hat{u}}_t$

Intuitive understanding As $\hat{u}_t$ is tangential vector along the path, its derivative quantifies how much the path is turning at a given instant.

Mathematical derivation

Given distance s along the path,

$$\dot{\hat{u}}_t = \frac{d\hat{u}_t}{dt} = \frac{ds}{dt}\frac{d\hat{u}_t}{ds} = v\frac{d\hat{u}_t}{ds}$$

Next investigate $\frac{d\hat{u}_t}{ds}$

$\frac{d\hat{u}_t}{ds}$ is the rate at which direction of velocity is changing with respect to the path. Denote magnitude of $\frac{d\hat{u}_t}{ds}$ by curvature k.

$$k(s) = \left|\frac{d\hat{u}_t}{ds}\right|$$

Further,

$$\hat{u}_t \circ \hat{u}_t = 1$$

Therefore,

$$\frac{d\hat{u}_t}{ds} \circ \hat{u}_t + \hat{u}_t \circ \frac{d\hat{u}_t}{ds} = 0$$

$$2\frac{d\hat{u}_t}{ds} \circ \hat{u}_t = 0$$

Therefore, $\frac{d\hat{u}_t}{ds}$ is perpendicular to $\hat{u}_t$.

Define $\hat{u}_n$ as (only when $k(s) \neq 0$

$$\hat{u}_n = \frac{d\hat{u}_t/ds}{|d\hat{u}_t/ds|} = \frac{1}{k(s)}\frac{d\hat{u}_t}{ds}$$

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Rewrite, $k(s) = 1/\rho(s)$

$$\hat{u}_n = \frac{1}{k(s)}\frac{d\hat{u}_t}{ds} = \rho(s)\frac{d\hat{u}_t}{ds}$$

Rearranging

$$\frac{d\hat{u}_t}{ds} = \frac{1}{\rho(s)}\hat{u}_n$$

If path is expressed as $y = f(x)$, then radius of curvature is given by

$$\rho(x) = \frac{[1 + (dy/dx)^2]^{\frac{3}{2}}}{|d^2y/dx^2|}$$

From before,

$$\vec{a} = \dot{v}\hat{u}_t + v\dot{\hat{u}}_t$$
$$= \dot{v}\hat{u}_t + \frac{v^2}{\rho}\hat{u}_n$$

Where

$$\rho(x) = \frac{[1 + (dy/dx)^2]^{\frac{3}{2}}}{|d^2y/dx^2|}$$

if $y = f(x)$.

Relative motion

$$\vec{r}_{B/A} = \vec{r}_{AB} = \vec{r}_B - \vec{r}_A$$
$$\vec{r}_{B/A} = (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j}$$
$$\vec{v}_{B/A} = \dot{\vec{r}}_{B/A} = (\dot{x}_B - \dot{x}_A)\hat{i} + (\dot{y}_B - \dot{y}_A)\hat{j}$$
$$\vec{a}_{B/A} = \dot{\vec{v}}_{B/A} = \ddot{\vec{r}}_{B/A} = (\dot{v}_{x,B} - \dot{v}_{x,A})\hat{i} + (\dot{v}_{y,B} - \dot{v}_{y,A})\hat{j}$$

<img src = 'images/rel_motion.jpg'>

Relative motion

$$\vec{r}_B = \vec{r}_{B/A} + \vec{r}_A$$
$$\vec{v}_B = \vec{v}_{B/A} + \vec{v}_A$$
$$\vec{a}_B = \vec{a}_{B/A} + \vec{a}_A$$

Differentiating Geometric constraints: Example 1

$$y_A = Lcos(\theta)$$

$$x_B = Lsin(\theta)$$

<img src = 'images/wall_link.jpg'>

Position

$$y_A = Lcos(\theta)$$

$$x_B = Lsin(\theta)$$

Velocity

Taking derivative gives,

$$v_A = -L\dot{\theta}sin(\theta)$$

$$v_B = L\dot{\theta}cos(\theta)$$

Acceleration

Taking derivative gives,

$$v_A = -\frac{d(L\dot{\theta}sin(\theta))}{dt} = -L\ddot{\theta}sin(\theta) - L\dot{\theta}^2 cos(\theta)$$

$$v_B = \frac{d(L\dot{\theta}cos(\theta))}{dt} = L\ddot{\theta}cos(\theta) - L\dot{\theta}^2 sin(\theta)$$

Constraint,

$$L^2 = x_B^2 + y_A^2$$

Taking derivative gives,

$$0 = 2x_B\dot{x}_B + 2\dot{y}_A y_A$$

Differentiating Geometric constraints: Example 2

<img src = 'images/pulley.jpg'>

Length of the rope is fixed,

Therefore,

$$L = AB + per(G) + CD + per(H) + EF$$

$$L = (y_P - GI) + per(G) + (y_P - GI - JH) + per(H) + (y_Q - JH)$$

Position

$$L = 2y_P + y_Q + per(G) - 2GI - 2JH + per(H)$$

Velocity

Taking derivative gives ,

$$0 = 2v_P + v_Q$$

Acceleration

Taking derivative gives ,

$$0 = 2a_P + a_Q$$

In [ ]:

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