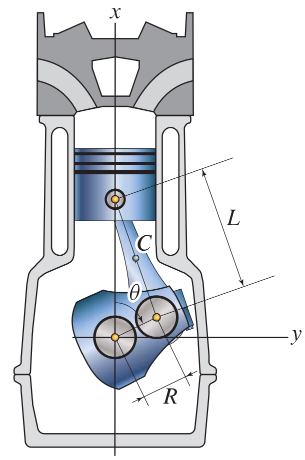
Problem 2.37

Point C is a point on the connecting rod of a mechanism called a *slider-crank*. The x and y coordinates of C can be expressed as follows: $x_C = R\cos\theta + \frac{1}{2}\sqrt{L^2 - R^2\sin^2\theta}$ and $y_C = (R/2)\sin\theta$, where θ describes the position of the crank. The crank rotates at a constant rate such that $\theta = \omega t$, where t is time.

Find expressions for the velocity, speed, and acceleration of C as functions of the angle θ and the parameters, R, L, and ω.



Solution

Using the coordinate system and expressions given in the problem statement, the position of point C can be expressed as as a function of θ as follows:

$$\vec{r}_C = x_C\,\hat{\imath} + y_C\,\hat{\jmath} = \left(R\cos\theta + \tfrac{1}{2}\sqrt{L^2 - R^2\sin^2\theta}\right)\hat{\imath} + \left(\tfrac{1}{2}R\sin\theta\right)\hat{\jmath}. \tag{1}$$

The velocity is the time derivative of the position. Hence, differentiating Eq. (1) with respect to time and using the chain rule, we have

$$\vec{v}_C = \frac{-\dot{\theta}R}{2}\left(2\sin\theta + \frac{R\sin\theta\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}}\right)\hat{\imath} + \frac{\dot{\theta}R}{2}\cos\theta\,\hat{\jmath}. \tag{2}$$

Since $\theta = \omega t$ and that therefore $\dot{\theta} = \omega$, we can rewrite Eq. (2) as

$$\boxed{\vec{v}_C = \frac{-\omega R}{2}\left(2\sin\theta + \frac{R\sin\theta\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}}\right)\hat{\imath} + \frac{\omega R}{2}\cos\theta\,\hat{\jmath}.} \tag{3}$$

The speed is now found by taking the magnitude of the velocity vector. Using Eq. (3), this gives

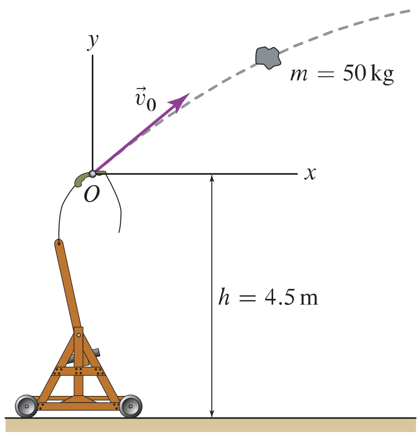
$$\boxed{v_C = \frac{\omega R}{2}\sqrt{4\sin^2\theta + \frac{4R\sin^2\theta\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}} + \frac{R^2\sin^2\theta\cos^2\theta}{L^2 - R^2\sin^2\theta} + \cos^2\theta}.} \tag{4}$$

The acceleration is found by taking the derivative of the velocity. Hence, differentiating Eq. (3) with respect to time, using the chain rule and recalling that $\dot{\theta} = \omega$, we have

$$\boxed{\vec{a}_C = \frac{-\omega^2 R}{2}\left[2\cos\theta + \frac{R(\cos^2\theta - \sin^2\theta)}{\sqrt{L^2 - R^2\sin^2\theta}} + \frac{R^3\cos^2\theta\sin^2\theta}{(L^2 - R^2\sin^2\theta)^{3/2}}\right]\hat{\imath} - \frac{\omega^2 R}{2}\sin\theta\,\hat{\jmath}.} \tag{5}$$

Problem 2.113

A trebuchet releases a rock with mass $m = 50\,\text{kg}$ at the point O. The initial velocity of the projectile is $\vec{v}_0 = (45\,\hat{\imath} + 30\,\hat{\jmath})\,\text{m/s}$. If one were to model the effects of air resistance via a drag force directly proportional to the projectile's velocity, the resulting accelerations in the x and y directions would be $\ddot{x} = -(\eta/m)\dot{x}$ and $\ddot{y} = -g - (\eta/m)\dot{y}$, respectively, where g is the acceleration of gravity and $\eta = 0.64\,\text{kg/s}$ is a viscous drag coefficient. Find an expression for the trajectory of the projectile.



Solution

We can integrate the x and the y components of acceleration to obtain the x and y displacement as a function of time. The problem states that $\ddot{x} = -(\eta/m)\dot{x}$. Then, recalling that $\ddot{x} = \frac{d\dot{x}}{dt}$, we can write

$$-\frac{\eta}{m}\dot{x} = \frac{d\dot{x}}{dt} \quad\Rightarrow\quad -\frac{\eta}{m}\,dt = \frac{d\dot{x}}{\dot{x}} \quad\Rightarrow\quad \int_{(v_0)_x}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\int_0^t \frac{\eta}{m}\,dt \quad\Rightarrow\quad \dot{x} = (v_0)_x e^{-\frac{\eta}{m}t}, \tag{1}$$

where $(v_0)_x$ is the x component of the velocity of the projectile at $t = 0$. Next, we recall that $\dot{x} = dx/dt$. So, using the last of Eqs. (1) we have

$$\frac{dx}{dt} = (v_0)_x e^{-\frac{\eta}{m}t} \quad\Rightarrow\quad dx = (v_0)_x e^{-\frac{\eta}{m}t}\,dt$$

$$\Rightarrow\quad \int_0^x dx = (v_0)_x \int_0^t e^{-\frac{\eta}{m}t}\,dt \quad\Rightarrow\quad x = \frac{m(v_0)_x}{\eta}\left(1 - e^{-\frac{\eta}{m}t}\right). \tag{2}$$

We can now repeat these steps starting with the acceleration in the y direction. Doing so, we have

$$-\int_{(v_0)_y}^{\dot{y}} \frac{d\dot{y}}{g + (\eta/m)\,\dot{y}} = \int_0^t dt \quad\Rightarrow\quad \dot{y} = \frac{mg}{\eta}\left(e^{-\frac{\eta}{m}t} - 1\right) + (v_0)_y e^{-\frac{\eta}{m}t} \tag{3}$$

where $(v_0)_y$ is the y component of the velocity of the projectile at $t = 0$. Integrating Eq. (3) again with respect to time, we obtain

$$y = \left(\frac{m^2 g}{\eta^2} - \frac{mgt}{\eta} + \frac{m}{\eta}(v_0)_y\right) - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right) e^{-\frac{\eta}{m}t}. \tag{4}$$

From Eq. (2) we find

$$e^{-\frac{\eta}{m}t} = \left(1 - \frac{\eta x}{m(v_0)_x}\right) \quad\Rightarrow\quad t = -\frac{m}{\eta}\ln\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{5}$$

Substituting the last of Eqs. (5) into Eq. (4) and recalling that $m = 50\,\text{kg}$, $(v_0)_x = 45\,\text{m/s}$, $(v_0)_y = 30\,\text{m/s}$ and $\eta = 0.64\,\text{kg/s}$, we obtain

$$\boxed{y = \left[59.88\times 10^3 \ln\left(1 - 2.844\times 10^{-4} x\right) + 17.70x\right]\,\text{m}.} \tag{6}$$

Problem 3.74

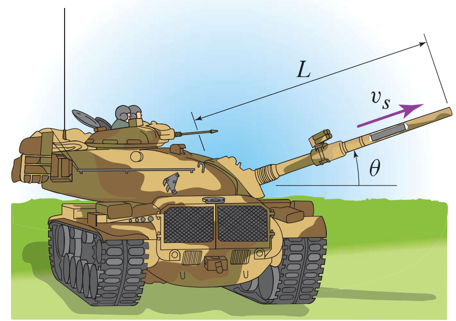
The cutaway of the gun barrel shows a projectile moving through the barrel. If the projectile's exit speed is $v_s = 1675\,\text{m/s}$ (relative to the barrel), the projectile's mass is 18.5 kg, the length of the barrel is $L = 4.4\,\text{m}$, the acceleration of the projectile down the gun barrel is constant, and θ is increasing at a constant rate of 0.18 rad/s, determine

(a) The acceleration of the projectile.

(b) The pressure force acting on the back of the projectile.

(c) The normal force on the gun barrel due to the projectile.

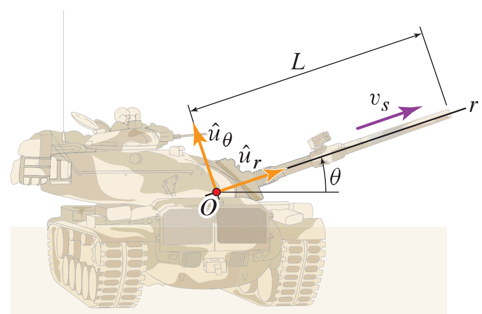
as the projectile leaves the gun, but while it is still in the barrel. Assume that the projectile exits the barrel when $\theta = 20°$, and ignore friction between the projectile and the barrel.



Solution

Part (a) of the problem consists of a purely kinematics question. By contrast, Parts (b) and (c) entail the computation of forces and the use of Newton's second law. We solve Part (a) first and then we will use the acceleration information found in this part as input to the solutions of Parts (b) and (c).

Part (a). We use the polar coordinate system with origin at O, as shown at the right. In this coordinate system, the acceleration of the projectile is



$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta. \tag{1}$$

Because $\ddot{r}$ is constant, we can compute $\ddot{r}$ using constant acceleration equations, i.e., $\dot{r}^2 = \dot{r}_0^2 + 2\ddot{r}(r - r_0)$. Letting $r_0 = 0$, since $\dot{r}_0 = 0$ (the projectile start from rest), and since we know that $\dot{r} = v_s$ for $r = L$, we can write

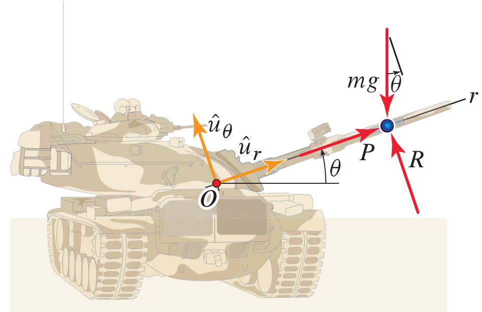
$$v_s^2 = 2\ddot{r}(2L - 0) \quad \Rightarrow \quad \ddot{r} = v_s^2/(2L). \tag{2}$$

Next, we observe that $\dot{\theta}$ is given and constant, so that $\ddot{\theta} = 0$. In summary, at $r = L$, Eq. (1) simplifies to

$$\vec{a} = \left(\frac{v_s^2}{2L} - L\dot{\theta}^2\right)\hat{u}_r + 2v_s\dot{\theta}\,\hat{u}_\theta. \tag{3}$$

Recalling that $v_s = 1675\,\text{m/s}$, $L = 4.4\,\text{m}$, and $\dot{\theta} = 0.18\,\text{rad/s}$, the expression in Eq. (3) can be evaluated to obtain

$$\boxed{\vec{a} = (318.8\times 10^3\,\hat{u}_r + 603.0\,\hat{u}_\theta)\,\text{m/s}^2.}$$



Parts (b) and (c). We model the projectile as a particle subject only to its own weight mg, the force P due to the pressure acting on the back of the shell, and the reaction R between the shell and the barrel.

Balance Principles. Applying Newton's second law, we have

$$\sum F_r\colon \quad P - mg\sin\theta = ma_r, \tag{4}$$

$$\sum F_\theta\colon \quad R - mg\cos\theta = ma_\theta. \tag{5}$$

Force Laws. All forces have been accounted for on the FBD and no additional force laws are necessary.

Kinematic Equations. Recalling that the acceleration of the projectile for $r = L$ has been found in Eq. (3), the kinematic equations for this problem are

$$a_r = \frac{v_s^2}{2L} - L\dot{\theta}^2 \quad \text{and} \quad a_\theta = 2v_s\dot{\theta}. \tag{6}$$

Computation. Substituting Eqs. (6) into Eqs. (4) and (5) we obtain

$$P - mg\sin\theta = m\left(\frac{v_s^2}{2L} - L\dot{\theta}^2\right) \quad \text{and} \quad R - mg\cos\theta = 2mv_s\dot{\theta}. \tag{7}$$

Equations (7) form a system of two equations in the two unknowns P and R, whose solution is

$$P = m\left(\frac{v_s^2}{2L} + g\sin\theta - L\dot{\theta}^2\right) \quad \text{and} \quad R = m\big(2v_s\dot{\theta} + g\cos\theta\big). \tag{8}$$

Recalling that $m = 18.5\,\text{kg}$, $v_s = 1675\,\text{m/s}$, $L = 4.4\,\text{m}$, $\dot{\theta} = 0.18\,\text{rad/s}$, $g = 9.81\,\text{m/s}^2$, and $\theta = 20^\circ$, the expression in Eq. (3) can be evaluated to obtain

$$\boxed{P = 5.898\times 10^6\,\text{N} \quad \text{and} \quad R = 11.33\times 10^3\,\text{N}.}$$